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A solution to the SU(n) external state labelling problem based upon a U($n-1, n-1$) group: I. General theory

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Abstract. The state labelling problem arising in the reduction of the direct product of a p positive-row U(n) irreducible representation $[h_1 \ldots h_p \dot{0}]$ with a q negative-row one $[\dot{0} -h'_q \ldots -h'_1]$ into a sum of mixed U(n) irreducible representations $[k_1 \ldots k_p \dot{0} - k'_q \ldots - k'_1]$ is solved by using the complementarity between U(n) and U(p, q) within some positive discrete series irreducible representations of U(pn, qn). This complementarity enables us to analyse the problem in terms of the group chain $\mathrm{U}(p, q) \supset \mathrm{U}(p) \times \mathrm{U}(q)$ instead of $\mathrm{U}(n) \times \mathrm{U}(n) \supset \mathrm{U}(n)$. For the most general SU(n) irreducible representations corresponding to $p = q = n-1$, the relevant group chain is therefore $\mathrm{U}(n-1, n-1) \supset \mathrm{U}(n-1) \times \mathrm{U}(n-1)$. In such a case, the additional labels include those of an intermediate U($n-1$) irreducible representation $[h''_1 \ldots h''_{n-1}]$, as well as the additional labels solving the state labelling problems for the products $[k_1 \ldots k_{n-1}] \times [h''_1 \ldots h''_{n-1}]$ and $[k'_1 \ldots k'_{n-1}] \times [h''_1 \ldots h''_{n-1}]$ of U($n-1$) irreducible representations. Hence the proposed solution reflects in a direct way the operation of King's branching rule for the chain $\mathrm{U}(n) \times \mathrm{U}(n) \supset \mathrm{U}(n)$, supplemented, whenever necessary, with King's modification rule.

1. Introduction

It is well known that the SU(n) external state labelling problem is equivalent to the internal state labelling problem for the chain $\mathrm{U}(n) \times \mathrm{U}(n) \supset \mathrm{U}(n)$ when only ($n-1$)-row U(n) irreducible representations (irreps) are considered. Over the past 25 years, the latter has been a topic of considerable research interest. Since the pioneering work of Moshinsky (1962, 1963), Baird and Biedenharn (1964, 1965) and Hecht (1965), various solutions have been discussed in connection with either non-orthonormal or orthonormal bases.

Among the former, one finds the solution initially proposed by Moshinsky (1962, 1963) and further developed by Brody *et al* (1965). This solution, based upon the elementary permissible diagram (EPD) method (Moshinsky and Syamala Devi 1969, Sharp and Lam 1969), leads to factorised, and hence highly tractable, bases, whose characterisation is completed by the exponents of some of their factors.

Among the latter, one finds the Baird and Biedenharn (1964, 1965) canonical solution, further studied by Biedenharn and co-workers (Biedenharn *et al* 1985 and

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references quoted therein). This approach uses SU(n) irreducible unit tensor operators, whose labelling is completed by an operator pattern. The canonical solution has been shown to be endowed with many nice structural properties. In the SU(3) case, it has recently been given a global algebraic formulation in the framework of an SO(6, 2) model (Biedenharn and Flath 1984, Bracken and MacGibbon 1984, Deenen and Quesne 1986); it has also led to a practical algorithm for the calculation of SU(3) Wigner and Racah coefficients (Draayer and Akiyama 1973).

The motivation for still another contribution to this much debated subject comes from a recent series of papers (Deenen and Quesne 1983, Quesne 1984, 1985b), where new solutions to the $U(n) \supset O(n)$ and $U(n) \supset USp(n)$ state labelling problems were proposed for the d-row U(n) irreps. These solutions embody substantial group theoretical information since they clearly exhibit how the internal state labelling problem for $U(n) \supset O(n)$ or $U(n) \supset USp(n)$ can be reduced to the external state labelling problem for U(d), according to Littlewood's (1950) branching rule, supplemented, whenever necessary, with Newell's (1951) modification rules (see also King 1971). In their derivation, a key role is played by the complementarity relationship (Moshinsky and Quesne 1970, Howe 1979) between the group chains $U(n) \supset O(n)$ and $Sp(2d, R) \supset U(d)$, on the one hand, and $U(n) \supset USp(n)$ and $SO^*(2d) \supset U(d)$ on the other hand (Moshinsky and Quesne 1971, Gross and Kunze 1977, Kashiwara and Vergne 1978, Gelbart 1979, Quesne 1985a).

Whether a similar solution based upon complementarity is possible for the SU(n) external state labelling problem is the question we will positively answer within the present series of papers. Although both the EPD and the Baird and Biedenharn solutions use complementarity properties, the former between the chains $U(n) \times U(n) \supset U(n)$ and $U(2n-2) \supset U(n-1) + U(n-1)$ and the latter between both U(n) factors in the sequence $U(n^2) \supset U(n) \times U(n) \supset U(n)$, here we have in mind a different complementarity: it relates the chain under consideration, $U(n) \times U(n) \supset U(n)$, with another one involving a non-compact group, as in our previous work.

The starting point of our analysis will be the recently reviewed (King and Wybourne 1985, Quesne (1986) complementarity between U(n) and U(p, q) within some positive discrete series irreps of U(pn, qn), characterised by a single label $[\rho]$ (Gross and Kunze 1977, Kashiwara and Vergne 1978, Gelbart 1979). The U(n) irreps appearing in the reduction of the U(pn, qn) irreps $[\rho]$ are mixed irreps, specified by $a(\leqslant p)$ positive and $b(\leqslant q)$ negative labels (Flores 1967, Flores and Moshinsky 1967, King 1970). We shall henceforth refer to such irreps as a positive-row and b negative-row ones.

We shall prove that the state labelling problem arising in the reduction of the direct product of p positive-row with q negative-row U(n) irreps can be solved by considering the group chain $U(p, q) \supset U(p) \times U(q)$, complementary with respect to $U(n) \times U(n) \supset U(n)$ when dealing with such irreps. For arbitrary SU(n) irreps, our solution to the SU(n) external state labelling problem is therefore based upon the sequence of groups $U(n-1, n-1) \supset U(n-1) \times U(n-1)$.

In § 2, we review the SU(n) external state labelling problem for the product of p positive-row with q negative-row irreps. In § 3, assuming $p+q \leqslant n$, we analyse such a problem in terms of the two complementary chains $U(n) \times U(n) \supset U(n)$ and $U(p, q) \supset U(p) \times U(q)$. In § 4, we construct bases of the U(n) subgroup scalar irreps and, in § 5, we use them to solve the SU(n) external state labelling problem for arbitrary irreps of the U(n) subgroup in the case where $p+q \leqslant n$. In § 6, we extend our solution to the case where $p+q > n$. Finally, § 7 contains the conclusion.

2. The SU(n) external state labelling problem

The SU(n) external state labelling problem can be discussed in terms of the group chain

$$\mathrm{U}(n)\times\mathrm{U}(n)\supset\mathrm{U}(n) \tag{2.1}$$

by declaring an equivalence relation on U(n) irreps:

$$[h_1+c, h_2+c, \ldots, h_n+c]\sim[h_1h_2\ldots h_n] \tag{2.2}$$

for c a finite integer. Here $h_1, h_2, \ldots, h_n$ are any (positive, zero or negative) integers subject to the conditions $h_1\geqslant h_2\geqslant\ldots\geqslant h_n$.

Let us realise the generators of the first U(n) group in equation (2.1) in terms of pn boson creation and annihilation operators η_{is}, ξ_{is}, $i=1,\ldots,p$, $s=1,\ldots,n$, as follows:

$$\mathscr{P}'_{st}=\tfrac{1}{2}\sum_{i=1}^{p}(\eta_{is}\xi_{it}+\xi_{it}\eta_{is})=\sum_{i=1}^{p}\eta_{is}\xi_{it}+\tfrac{1}{2}p\delta_{st}\qquad s,t=1,\ldots,n \tag{2.3}$$

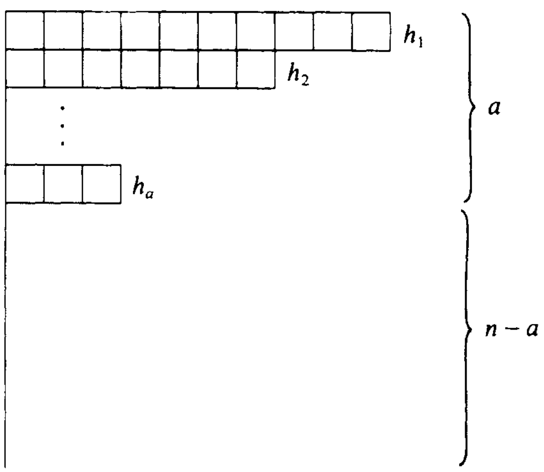
where $p\leqslant n-1$. The operators $\mathscr{P}'_{st}$ differ from the usual ones by a constant $\frac{1}{2}p\delta_{st}$, whose introduction will later on prove convenient. It affects neither their commutation relations, nor their hermiticity properties, given by respectively

$$[\mathscr{P}'_{st}, \mathscr{P}'_{s't'}]=\delta_{ts'}\mathscr{P}'_{st'}-\delta_{st'}\mathscr{P}'_{s't} \tag{2.4}$$

and

$$(\mathscr{P}'_{st})^{+}=\mathscr{P}'_{ts}. \tag{2.5}$$

In such a realisation, the creation operators η_{is} form a contravariant U(n) vector, while the annihilation operators ξ_{is} form a covariant one. The U(n) irreps are characterised by their highest weight $\{h_1+\frac{1}{2}p, \ldots, h_a+\frac{1}{2}p, (\frac{1}{2}p)^{n-a}\}$, where $h_1,\ldots,h_a$ are a positive integers such that $a\leqslant p$ and $h_1\geqslant\ldots\geqslant h_a$. We denote them in short by $[h_1\ldots h_a\dot{0}]$, where the dot over the zero means that it appears $n-a$ times (in general, a dot over a numeral implies that this numeral is repeated as often as necessary). They can be represented by a Young diagram of the form



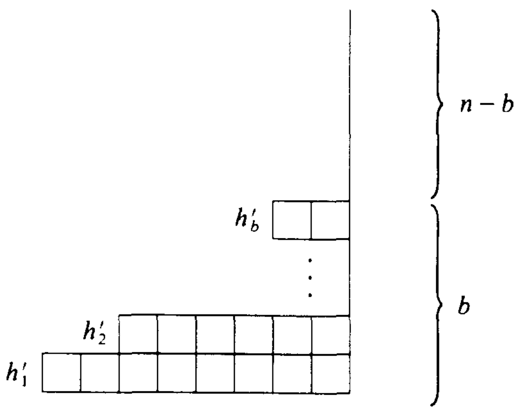
(2.6)

where, counting from top to bottom, on the right of the vertical line there are h_s boxes in the sth row for $s=1,\ldots,a$ and no boxes in the remaining $n-a$ rows.

For the generators of the second U(n) group in equation (2.1), let us choose a different realisation in terms of qn pairs of boson creation and annihilation operators η_{is}, ξ_{is}, $i=p+1,\ldots,d$, $s=1,\ldots,n$, namely

$$\mathscr{P}''_{st}=-\tfrac{1}{2}\sum_{i=p+1}^{d}(\eta_{it}\xi_{is}+\xi_{is}\eta_{it})=-\sum_{i=p+1}^{d}\eta_{it}\xi_{is}-\tfrac{1}{2}q\delta_{st}\qquad s,t=1,\ldots,n \tag{2.7}$$

where $q\leqslant n-1$ and $d=p+q$. It is straightforward to check that the operators $\mathscr{P}''_{st}$ satisfy the same commutation relations and hermiticity properties as the operators $\mathscr{P}'_{st}$. In the realisation (2.7), the creation operators η_{is} form a covariant U(n) vector, while the annihilation operators ξ_{is} form a contravariant one. The corresponding U(n) irreps $\{(\frac{1}{2}q)^{n-b}, -h'_b-\frac{1}{2}q,\ldots, -h'_1-\frac{1}{2}q\}\equiv[\dot{0}-h'_q\ldots-h'_1]$ are characterised by b negative integers $-h'_b,\ldots,-h'_1$, such that $b\leqslant q$ and $h'_1\geqslant\ldots\geqslant h'_b$. They can be represented by a generalised Young diagram of the form (Flores 1967, Flores and Moshinsky 1967)



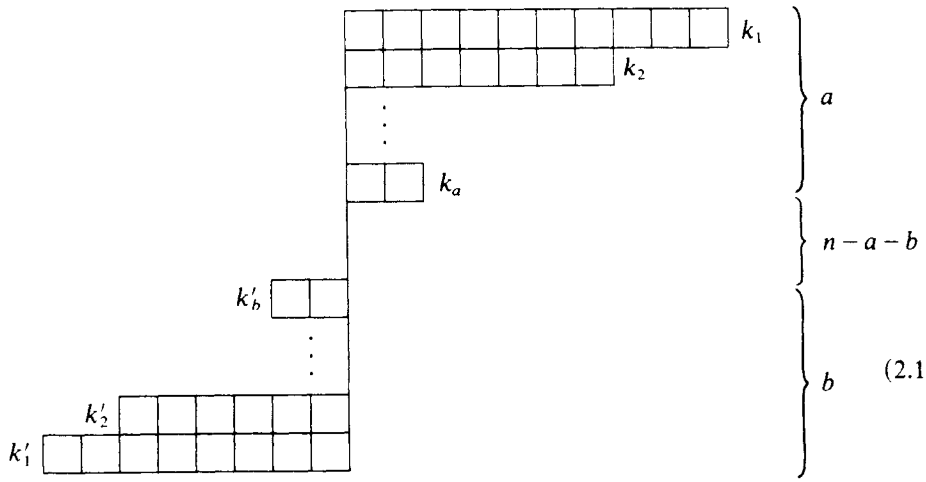
(2.8)

where, counting from bottom to top, on the left of the vertical line there are h'_s boxes in the sth row for $s=1,\ldots,b$, and no boxes in the remaining $n-b$ rows.

The U(n) subgroup of U(n)$\times$U(n) is generated by the operators

$$\mathscr{P}_{st}=\mathscr{P}'_{st}+\mathscr{P}''_{st}=\sum_{i=1}^{p}\eta_{is}\xi_{it}-\sum_{i=p+1}^{d}\eta_{it}\xi_{is}+\tfrac{1}{2}(p-q)\delta_{st}\qquad s,t=1,\ldots,n. \tag{2.9}$$

Whenever $p+q\leqslant n$, its irreps $\{k_1+\frac{1}{2}(p-q),\ldots, k_a+\frac{1}{2}(p-q), [\frac{1}{2}(p-q)]^{n-a-b}, -k'_b+\frac{1}{2}(p-q),\ldots, -k'_1+\frac{1}{2}(p-q)\}\equiv[k_1\ldots k_a\dot{0}-k'_b\ldots-k'_1]$ are specified by a positive and b negative integers, denoted by $k_1,\ldots,k_a$, and $-k'_b,\ldots,-k'_1$ respectively, and satisfying the conditions $a\leqslant p$, $b\leqslant q$, $k_1\geqslant\ldots\geqslant k_a$ and $k'_1\geqslant\ldots\geqslant k'_b$ (Flores 1967, Flores and Moshinsky 1967, King 1970). They can be represented by a generalised Young diagram of the form (Flores 1967, Flores and Moshinsky 1967)



(2.10)

where, counting from top to bottom, on the right of the vertical line there are k_s boxes in the sth row for $s = 1, \ldots, a$, and, counting from bottom to top, on the left of the vertical line there are k'_s boxes in the sth row for $s = 1, \ldots, b$, while there are no boxes in the remaining $n-a-b$ rows.

The reduction of the product of a p positive-row U(n) irrep $[h_1 \ldots h_p \dot{0}]$ with a q negative-row one $[\dot{0} - h'_q \ldots - h'_1]$ into a sum of U(n) irreps,

$$[h_1 \ldots h_p \dot{0}] \times [\dot{0} - h'_q \ldots - h'_1] \downarrow \sum_{\substack{k_1 \ldots k_p \\ k'_1 \ldots k'_q}} m_{[h_1 \ldots h_p \dot{0}][\dot{0} - h'_q \ldots - h'_1][k_1 \ldots k_p \dot{0} - k'_q \ldots - k'_1]}$$
$$\times [k_1 \ldots k_p \dot{0} - k'_q \ldots - k'_1] \tag{2.11}$$

where $m_{\rho\sigma\tau}$ denotes the multiplicity of τ in $\rho \times \sigma$ and $k_{a+1} = \ldots = k_p = k'_{b+1} = \ldots = k'_q = 0$ for some $a \leqslant p$ and $b \leqslant q$, can be determined by using a reformulation of the Littlewood-Richardson rule (1934), valid for mixed irreps (Flores 1967, Flores and Moshinsky 1967, King 1970). Alternatively, one can apply the following result for the multiplicities (King 1970):

$$m_{[h_1 \ldots h_p \dot{0}][\dot{0} - h'_q \ldots - h'_1][k_1 \ldots k_p 0 - k'_q \ldots - k'_1]}$$
$$= \sum_{h^s_1 \ldots h^s_q} m_{[k_1 \ldots k_p][h^s_1 \ldots h^s_q \dot{0}][h_1 \ldots h_p]} m_{[k'_1 \ldots k'_q][h^s_1 \ldots h^s_q][h'_1 \ldots h'_q]} \tag{2.12}$$

where the summation is taken over q non-negative integers $h^s_1, \ldots, h^s_q$, subject to the conditions $h^s_1 \geqslant \ldots \geqslant h^s_q$, and the multiplicities on the right-hand side, referring to positive-row U(n) irreps, can be calculated by means of the standard Littlewood-Richardson rule. In equation (2.12), we have assumed that $q \leqslant p$. For SU(n) irreps, the validity of this hypothesis can always be ensured by using equation (2.1).

Whenever $p+q > n$, (2.11) and (2.12) are still valid. However, on the right-hand side of (2.11), there may appear non-standard U(n) irreps, corresponding to inadmissible Young diagrams (diagrams with positive and negative blocks within the same row). Such non-standard irreps have to be converted into standard ones, associated with admissible Young diagrams (diagrams of type (2.10) with $a+b \leqslant n$), by using a modification rule (King 1971). In the cases where $n=3$, $p=q=2$, and $n=4$, $p=q=3$, for instance, the latter is

$$[k_1 k_2 - k'_2 - k'_1] = 0 \qquad \text{whenever } k_2 \text{ and } k'_2 \neq 0 \tag{2.13}$$

and

$$\begin{aligned}
&[k_1 k_2 k_3 - k_2' - k_1'] = [k_1 k_2 - k_3' - k_2' - k_1'] = 0 \qquad && \text{whenever } k_3 \text{ or } k_3' \neq 0 \\
&[k_1 k_2 1 - 1 - k_2' - k_1'] = -[k_1 k_2 - k_2' - k_1'] && \\
&[k_1 k_2 k_3 - k_3' - k_2' - k_1'] = 0 && \text{for all other cases with } k_3 \text{ and } k_3' \neq 0
\end{aligned} \tag{2.14}$$

respectively. In general, for arbitrary n, p and q values, the standard $\mathrm{U}(n)$ irreps one is left with are of the type $[k_1 \ldots k_{n-q+\sigma} - k'_{q-\sigma} \ldots - k'_1]$, where $k_1, \ldots, k_{n-q+\sigma}, k'_1, \ldots, k'_{q-\sigma}$ are non-negative integers, subject to the conditions $k_1 \geqslant \ldots \geqslant k_{n-q+\sigma}$ and $k'_1 \geqslant \ldots \geqslant k'_{q-\sigma}$, and σ is any integer belonging to the set $\{0, 1, \ldots, p+q-n\}$. Since the modification rule complicates the reduction procedure quite a lot, we shall leave the discussion of the case $p+q>n$ until § 6 and from now on assume $p+q \leqslant n$.

To conclude the present section, let us determine the number of missing labels in the reduction of $\mathrm{U}(n) \times \mathrm{U}(n)$ irreps of the type $[h_1 \ldots h_p \dot{0}] \times [\dot{0} - h'_q \ldots - h'_1]$. Since the labels of a generic $\mathrm{U}(n)$ irrep $[k_1 \ldots k_p \dot{0} - k'_q \ldots - k'_1]$ contained in such a product are linked by the relation

$$\sum_\alpha k_\alpha - \sum_\beta k'_\beta = \sum_\alpha h_\alpha - \sum_\beta h'_\beta \tag{2.15}$$

where α and β go from 1 to p, and 1 to q, respectively, there are only $p+q-1$ independent labels. On the other hand, the number of internal labels, necessary to completely specify the states of a degenerate $\mathrm{U}(n)$ irrep with $n-p$ vanishing labels, is equal to $pn - \frac{1}{2}p(p+1)$ (Seligman and Sharp 1983). Hence, for a generic $\mathrm{U}(n)$ irrep $[k_1 \ldots k_p \dot{0} - k'_q \ldots - k'_1]$, the number of missing labels is given by

$$\begin{aligned}
&[pn - \tfrac{1}{2}p(p+1)] + [qn - \tfrac{1}{2}q(q+1)] - (p+q-1) - [(p+q)n - \tfrac{1}{2}(p+q)(p+q-1)] \\
&\qquad = (p-1)(q-1).
\end{aligned} \tag{2.16}$$

In the next section, we shall define a complementary chain with respect to (2.1) and reformulate the $\mathrm{SU}(n)$ external state labelling problem in terms of this new sequence of groups.

3. The SU(n) external state labelling problem in terms of two complementary chains

Let us consider the operators $\mathbb{P}_{is,jt}$, $i, j = 1, \ldots, d$, s, $t = 1, \ldots, n$, defined by

$$\mathbb{P}_{is,jt} = \begin{cases} \mathbb{E}_{is,jt} & \text{if } i, j = 1, \ldots, p \\ \mathbb{E}_{jt,is} & \text{if } i, j = p+1, \ldots, d \\ \mathbb{D}^\dagger_{is,jt} & \text{if } i = 1, \ldots, p \text{ and } j = p+1, \ldots, d \\ \mathbb{D}_{is,jt} & \text{if } i = p+1, \ldots, d \text{ and } j = 1, \ldots, p \end{cases} \tag{3.1}$$

where

$$\begin{aligned}
&\mathbb{D}^\dagger_{is,jt} = \eta_{is}\eta_{jt} \qquad \mathbb{D}_{is,jt} = \xi_{is}\xi_{jt} \\
&\mathbb{E}_{is,jt} = \tfrac{1}{2}(\eta_{is}\xi_{jt} + \xi_{jt}\eta_{is}) = \eta_{is}\xi_{jt} + \tfrac{1}{2}\delta_{ij}\delta_{st}.
\end{aligned} \tag{3.2}$$

They satisfy the hermiticity properties

$$(\mathbb{P}_{is,jt})^\dagger = \mathbb{P}_{jt,is} \tag{3.3}$$

and the commutation relations

$$[\mathbb{P}_{is,jt}, \mathbb{P}_{i's',j't'}] = g_{jt,i's'}\mathbb{P}_{is,j't'} - g_{j't',is}\mathbb{P}_{i's',jt} \tag{3.4}$$

where

$$g_{is,jt} = \varepsilon_i \delta_{ij} \delta_{st} \tag{3.5}$$

and

$$\varepsilon_i = \begin{cases} +1 & \text{if } i = 1, \ldots, p \\ -1 & \text{if } i = p+1, \ldots, d. \end{cases} \tag{3.6}$$

Hence they generate a U(pn, qn) group (Quesne 1986). Two U(pn, qn) subgroup chains (King and Wybourne 1985) play an important role in the discussion of the SU(n) external state labelling problem.

The first chain is

$$\begin{array}{ccccccccc}
 & & & & [h_1 \ldots h_p \dot{0}] & & [\dot{0} - h'_q \ldots - h'_1] & [h_1 \ldots h_p] & [h'_1 \ldots h'_q] \\
U(pn, qn) & \supset & \mathrm{U}(pn) \times \mathrm{U}(qn) & \supset & [\mathrm{U}(n) & \times & \mathrm{U}(n)] \times & [\mathrm{U}(p) & \times \mathrm{U}(q)] \\
{[\rho]} & & [N\dot{0}] \quad [N'\dot{0}] & & & \cup & & & \\
 & & & & & \mathrm{U}(n) & & & \\
 & & & & & [k_1 \ldots k_p \dot{0} - k'_q \ldots - k'_1] & & &
\end{array} \tag{3.7}$$

where U$(pn) \times$ U(qn) is the maximal compact subgroup of U(pn, qn) and U(pn), U(qn) are generated by the operators $\mathbb{E}_{is,jt}$, $i, j = 1, \ldots, p$, $s, t = 1, \ldots, n$, and $\mathbb{E}_{is,jt}$, $i, j = p+1, \ldots, d$, $s, t = 1, \ldots, n$, respectively. To define the U(n) and U(p) subgroups of U(pn), we contract as usual its generators over index i or s, thereby obtaining the operators $\mathscr{P}'_{st}$, $s, t = 1, \ldots, n$, defined in (2.3), and the operators $E_{ij} = \Sigma_s \mathbb{E}_{is,js}$, $i, j = 1, \ldots, p$. For the U(n) and U(q) subgroups of U(qn), we proceed by the same way, then apply to the U(n) generators the automorphism of the u(n) algebra

$$\sum_{i=p+1}^{d} \mathbb{E}_{is,it} \rightarrow -\sum_{i=p+1}^{d} \mathbb{E}_{it,is} \tag{3.8}$$

thereby getting the operators $\mathscr{P}''_{st}$, $s, t = 1, \ldots, n$, defined in (2.7), and the operators $E_{ij} = \Sigma_s \mathbb{E}_{is,js}$, $i, j = p+1, \ldots, d$. It follows that the chain (2.1) belongs to the sequence of groups (3.7).

In the realisation (3.1), the U(pn, qn) group has only positive discrete series irreps $[\rho]$, specified by a single (positive, zero or negative) integer ρ, related to the eigenvalue $\rho + \frac{1}{2}(p-q)n$ of the first-order Casimir operator (Quesne 1986)

$$\mathbb{G}_1 = \sum_{is} \varepsilon_i \mathbb{P}_{is,is}. \tag{3.9}$$

In (3.7), we have indicated below or above each U(pn, qn) subgroup the labels characterising its irreps contained in a given U(pn, qn) irrep $[\rho]$. The U(pn) and U(qn) irreps are denoted by their highest weight $\{N + \frac{1}{2}, (\frac{1}{2})^{pn-1}\} \equiv [N\dot{0}]$ and $\{N' + \frac{1}{2}, (\frac{1}{2})^{qn-1}\} \equiv [N'\dot{0}]$ respectively. Here N and N' are two non-negative integers satisfying the condition $N - N' = \rho$. Since U(p) and the first U(n) subgroup are complementary (Moshinsky and Quesne 1970) within any irrep $[N\dot{0}]$ of U(pn), their irreps are characterised by the same partition of N into p non-negative integers $h_1, \ldots, h_p$ (Moshinsky 1963). A similar property holds true for $\mathrm{U}(q)$ and the second $\mathrm{U}(n)$ subgroup because the automorphism (3.8) transforms an irrep $[h'_1 \ldots h'_q\dot{0}]$ of $\mathrm{U}(n)$ into $[\dot{0} - h'_q \ldots - h'_1]$ and therefore preserves the complementarity relation between $\mathrm{U}(q)$ and $\mathrm{U}(n)$ within the irreps $[N'\dot{0}]$ of $\mathrm{U}(qn)$.

The second relevant subgroup chain is

$$\begin{array}{ccccc} & [k_1 \ldots k_p;\, k'_1 \ldots k'_q] & & [k_1 \ldots k_p\dot{0} - k'_q \ldots - k'_1] & \\ \mathrm{U}(pn, qn) \supset & \mathrm{U}(p, q) & \times & \mathrm{U}(n) & \quad (3.10) \\ [\rho] & \cup & & & \\ & \mathrm{U}(p) \times \mathrm{U}(q) & & & \\ & [h_1 \ldots h_p][h'_1 \ldots h'_q] & & & \end{array}$$

where the $\mathrm{U}(n)$ and $\mathrm{U}(p) \times \mathrm{U}(q)$ groups are the same as in (3.7), i.e. are generated by the operators $\mathscr{P}_{st}$, $s, t = 1, \ldots, n$ and E_{ij}, $i, j = 1, \ldots, p$, or $i, j = p+1, \ldots, d$, respectively. The only group appearing in (3.10), and not present in (3.7), is $\mathrm{U}(p, q)$. Such a group is generated by the operators (Quesne 1986)

$$P_{ij} = \sum_s \mathbb{P}_{is,js} = \begin{cases} E_{ij} & \text{if } i, j = 1, \ldots, p \\ E_{ji} & \text{if } i, j = p+1, \ldots, d \\ D^\dagger_{ij} & \text{if } i = 1, \ldots, p \text{ and } j = p+1, \ldots, d \\ D_{ij} & \text{if } i = p+1, \ldots, d \text{ and } j = 1, \ldots, p \end{cases} \qquad (3.11)$$

where $D^\dagger_{ij}$, D_{ij} and E_{ij} are the contractions over s of the operators (3.2). The operators P_{ij} satisfy commutation relations similar to (3.4) with $g_{ij} = \varepsilon_i \delta_{ij}$ substituted for $g_{is,jt}$, and moreover they commute with $\mathscr{P}_{st}$.

In (3.10), we have indicated below or above each subgroup the labels characterising its irreps contained in a given irrep $[\rho]$ of $\mathrm{U}(pn, qn)$. The $\mathrm{U}(p, q)$ and $\mathrm{U}(n)$ groups are complementary within any irrep $[\rho]$, the branching rule for the latter being (Quesne 1986)

$$[\rho]\downarrow \sum_{\substack{k_1 \ldots k_p \\ k'_1 \ldots k'_q}} ([k_1 \ldots k_p;\, k'_1 \ldots k'_q] \times [k_1 \ldots k_p\dot{0} - k'_q \ldots - k'_1]) \qquad (3.12)$$

where the summation runs over all the partitions $[k_1 \ldots k_p]$ and $[k'_1 \ldots k'_q]$ into p or q non-negative integers, subject to the condition $\Sigma_\alpha k_\alpha - \Sigma_\beta k'_\beta = \rho$. The $\mathrm{U}(p, q)$ irreps are positive discrete series ones, specified by their lowest weight $\{k_p + \frac{1}{2}n, \ldots, k_1 + \frac{1}{2}n;\, k'_q + \frac{1}{2}n, \ldots, k'_1 + \frac{1}{2}n\} \equiv [k_1 \ldots k_p;\, k'_1 \ldots k'_q]$. The $\mathrm{U}(n)$ and $\mathrm{U}(p) \times \mathrm{U}(q)$ irreps are of course the same in (3.7) and (3.10).

Comparing (3.7) with (3.10), we obtain the following two chains of complementary groups:

$$\begin{array}{cccc} [h_1 \ldots h_p\dot{0}] & [\dot{0} - h'_q \ldots - h'_1] & & [k_1 \ldots k_p\dot{0} - k'_q \ldots - k'_1] \\ \mathrm{U}(n) \times & \mathrm{U}(n) & \supset & \mathrm{U}(n) \end{array} \qquad (3.13a)$$

$$\begin{array}{cccc} \mathrm{U}(p) \times & \mathrm{U}(q) & \subset & \mathrm{U}(p, q) \\ [h_1 \ldots h_p] & [h'_1 \ldots h'_q] & & [k_1 \ldots k_p;\, k'_1 \ldots k'_q] \end{array} \qquad (3.13b)$$

where we have written the pairs of complementary groups (and their respective irreps) one below the other.

Let us now consider the highest weight states (HWS) $P(\eta_{is})|0\rangle$ of all the equivalent $U(n)$ subgroup irreps characterised by $[k_1 \ldots k_p \dot{0} - k'_q \ldots - k'_1]$ and contained in an irrep $[h_1 \ldots h_p \dot{0}] \times [\dot{0} - h'_q \ldots - h'_1]$ of $U(n) \times U(n)$. Here $|0\rangle$ is the boson vacuum state and $P(\eta_{is})$ is a polynomial in the boson creation operators η_{is}, $i = 1, \ldots, d$, $s = 1, \ldots, n$. Such HWS are the simultaneous solutions of the system of equations:

$$\sum_s \eta_{is}\xi_{is} P(\eta_{is})|0\rangle = \begin{cases} h_i P(\eta_{is})|0\rangle & i = 1, \ldots, p \\ h'_{i-p} P(\eta_{is})|0\rangle & i = p+1, \ldots, d \end{cases} \tag{3.14a}$$

$$\sum_s \eta_{is}\xi_{js} P(\eta_{is})|0\rangle = 0 \qquad 1 \leqslant i < j \leqslant p,\ p+1 \leqslant i < j \leqslant d \tag{3.14b}$$

$$\left(\sum_{i=1}^{p} \eta_{is}\xi_{is} - \sum_{i=p+1}^{d} \eta_{is}\xi_{is} \right) P(\eta_{is})|0\rangle = \begin{cases} k_s P(\eta_{is})|0\rangle & s = 1, \ldots, p \\ 0 & s = p+1, \ldots, n-q \\ -k'_{n+1-s} P(\eta_{is})|0\rangle & s = n-q+1, \ldots, n \end{cases} \tag{3.14c}$$

$$\left(\sum_{i=1}^{p} \eta_{is}\xi_{it} - \sum_{i=p+1}^{d} \eta_{it}\xi_{is} \right) P(\eta_{is})|0\rangle = 0 \qquad 1 \leqslant s < t \leqslant n. \tag{3.14d}$$

From the complementarity of chains (3.13*a*) and (3.13*b*), it follows that the simultaneous solutions of (3.14) are also the HWS of all the equivalent $U(p) \times U(q)$ irreps characterised by $[h_1 \ldots h_p] \times [h'_1 \ldots h'_q]$ and contained in a $U(p, q)$ irrep $[k_1 \ldots k_p;\ k'_1 \ldots k'_q]$. In other words, the state labelling problems for both complementary chains are completely equivalent. If we denote by ω a set of $(p-1) \times (q-1)$ additional labels, distinguishing between repeated $U(n)$ irreps contained in a given $U(n) \times U(n)$ irrep—or equivalently between repeated $U(p) \times U(q)$ irreps contained in a given $U(p, q)$ irrep—the simultaneous solutions of (3.14) can be written as

$$\left| \begin{matrix} [k_1 \ldots k_p;\ k'_1 \ldots k'_q] & [h_1 \ldots h_p \dot{0}][\dot{0} - h'_q \ldots - h'_1] \\ \omega[h_1 \ldots h_p][h'_1 \ldots h'_q]; & \omega[k_1 \ldots k_p \dot{0} - k'_q \ldots - k'_1] \\ (\max) \quad (\max) & (\max) \end{matrix} \right\rangle = A_\omega P_\omega(\eta_{is})|0\rangle \tag{3.15}$$

where the right-(left-)hand part of the ket characterises the irreps of the chain (3.13*a*) ((3.13*b*)) and A_ω is some normalisation coefficient (whose dependence upon the irrep labels has not been indicated).

In § 5, we shall construct the set (3.15), thereby showing how the $(p-1) \times (q-1)$ additional labels ω can be chosen. For such purpose, we shall need the (unique) solution of (3.14) for the special case of a $U(n)$ subgroup scalar irrep that we shall determine in § 4.

4. The case of a U(*n*) subgroup scalar representation

Equation (2.12) shows that the $U(n)$ scalar irrep $[\dot{0}]$ is contained with multiplicity one in the $U(n) \times U(n)$ irreps $[h_1 \ldots h_q \dot{0}] \times [\dot{0} - h_q \ldots - h_1]$, while it does not appear in the remaining irreps. We shall henceforth denote the $U(n) \times U(n)$ irreps containing the irrep $[\dot{0}]$ of $U(n)$ by the symbol $[h^\backprime_1 \ldots h^\backprime_q \dot{0}] \times [\dot{0} - h^\backprime_q \ldots - h^\backprime_1]$.

According to (3.15), the unique solution of (3.14) corresponding to the scalar case can be written as

$$\left|\begin{matrix} [\dot{0};\dot{0}] & [h_1^s\dots h_q^s\dot{0}][\dot{0}-h_q^s\dots -h_1^s] \\ [h_1^s\dots h_q^s\dot{0}][h_1^s\dots h_q^s]; & [\dot{0}] \\ (\max) \quad (\max) & \end{matrix}\right\rangle = A^s P^s(\eta_{is})|0\rangle \tag{4.1}$$

no additional label ω being needed. Since the lowest weight state (LWS) of the $U(p,q)$ irrep $[\dot{0};\dot{0}]$ is the boson vacuum state (Quesne 1986) and the $U(p,q)$ generators D_{ij} annihilate this state while the operators E_{ij} reduce to the constants $\frac{1}{2}n\delta_{ij}$ when acting upon it, all the basis states of the $U(p,q)$ irrep $[\dot{0};\dot{0}]$ can be obtained from $|0\rangle$ by applying a polynomial in the remaining generators $D_{ij}^\dagger$. Hence, in (4.1), we may write

$$P^s(\eta_{is}) = P^s(D_{ij}^\dagger). \tag{4.2}$$

The explicit form of $P^s(D_{ij}^\dagger)$ can be found by solving (3.14) for the appropriate values of the quantum numbers. This is most easily done by applying the EPD method (Moshinsky and Syamala Devi 1969, Sharp and Lam 1969). There are q EPD, corresponding to the $U(n)\times U(n)$ irreps $[1^\beta\dot{0}]\times[\dot{0}(-1)^\beta]$, $\beta=1,\dots,q$, respectively. Their HWS can be written as $D^\dagger_{12\dots\beta,p+1\,p+2\dots p+\beta}|0\rangle$, where

$$D^\dagger_{12\dots\beta,p+1\,p+2\dots p+\beta} = \sum_\pi (-1)^\pi D^\dagger_{1,\pi(p+1)} D^\dagger_{2,\pi(p+2)}\dots D^\dagger_{\beta,\pi(p+\beta)} \tag{4.3}$$

and the summation is carried out over the $\beta!$ permutations of the indices $p+1, p+2,\dots,p+\beta$. In terms of them, the polynomial $P^s(D_{ij}^\dagger)$ is

$$P^s(D_{ij}^\dagger) = \prod_{\beta=1}^{q} (D^\dagger_{12\dots\beta,p+1\,p+2\dots p+\beta})^{h_\beta - h_{\beta+1}} \tag{4.4}$$

where $h_{q+1}=0$. It will be shown elsewhere (Quesne 1987) that the normalisation coefficient A^s appearing in (4.1) can be determined by using a coherent state representation of $U(p,q)$ and is given by

$$A^s = \left[\left(\prod_{\beta<\beta'}^{q} (h_\beta^s - h_{\beta'}^s + \beta' - \beta)\right)\left(\prod_{\beta=1}^{q} (n-\beta)!\right) \times \left(\prod_{\beta=1}^{q} (h_\beta^s + q - \beta)!(h_\beta^s + n - \beta)!\right)^{-1}\right]^{1/2}. \tag{4.5}$$

5. The general case for $p+q\leqslant n$

Let us consider (3.14), where $h_1,\dots,h_p$, $h'_1,\dots h'_q$, $k_1,\dots k_p$, and $k'_1,\dots,k'_q$ now assume arbitrary values compatible with (2.11) and (2.12) but p and q are still restricted by the condition $p+q\leqslant n$. Since their simultaneous solutions (3.15) belong to a $U(p,q)$ irrep $[k_1\dots k_p;\ k'_1\dots k'_q]$, let us first construct the carrier space of the latter, and then search for the $U(p)\times U(q)$ HWS it contains.

The LWS of a $U(p,q)$ irrep $[k_1\dots k_p;\ k'_1\dots k'_q]$ is at the same time the LWS of a $U(p)\times U(q)$ irrep $[k_1\dots k_p]\times[k'_1\dots k'_q]$. In the notations of (3.15), the HWS of the latter, which is also of highest weight in $U(n)$, can be written as

$$\left|\begin{matrix} [k_1\dots k_p;\ k'_1\dots k'_q] & [k_1\dots k_p\dot{0}][\dot{0}-k'_q\dots -k'_1] \\ [k_1\dots k_p][k'_1\dots k'_q]; & [k_1\dots k_p\dot{0}-k'_q\dots -k'_1] \\ (\max) \quad (\max) & (\max) \end{matrix}\right\rangle = \bar{A}\bar{P}(\eta_{is})|0\rangle \tag{5.1}$$

where no additional label ω is needed. The explicit form of $\bar{P}(\eta_{is})$ is given by (Quesne 1986)

$$\bar{P}(\eta_{is}) = \left(\prod_{\alpha=1}^{p} (\eta_{1\ldots\alpha,1\ldots\alpha})^{k_\alpha - k_{\alpha+1}} \right) \left(\prod_{\beta=1}^{q} (\eta_{p+1\ldots p+\beta, n-\beta+1\ldots n})^{k'_\beta - k'_{\beta+1}} \right) \tag{5.2}$$

where $k_{p+1} = k'_{q+1} = 0$ and $\eta_{i_1\ldots i_r, s_1\ldots s_r}$ is defined by a relation similar to (4.3). In (5.1), the normalisation coefficient $\bar{A}$ is equal to (Brody *et al* 1965)

$$\bar{A} = \left[\left(\prod_{\alpha<\alpha'}^{p} (k_\alpha - k_{\alpha'} + \alpha' - \alpha) \right) \left(\prod_{\alpha=1}^{p} (k_\alpha + p - \alpha)! \right)^{-1} \right]^{1/2}$$
$$\times \left[\left(\prod_{\beta<\beta'}^{q} (k'_\beta - k'_{\beta'} + \beta' - \beta) \right) \left(\prod_{\beta=1}^{q} (k'_\beta + q - \beta)! \right)^{-1} \right]^{1/2}. \tag{5.3}$$

From the HWS (5.1), we can generate all the basis states of the $U(p) \times U(q)$ irrep $[k_1 \ldots k_p] \times [k'_1 \ldots k'_q]$ by applying appropriate $U(p)$, $U(p-1), \ldots, U(2)$ and $U(q)$, $U(q-1), \ldots, U(2)$ lowering operators (Nagel and Moshinsky 1965). The resulting states

$$\left| \begin{matrix} [k_1 \ldots k_p; k'_1 \ldots k'_q] & [k_1 \ldots k_p \dot{0}][\dot{0} - k'_q \ldots - k'_1] \\ [k_1 \ldots k_p][k'_1 \ldots k'_q]; & [k_1 \ldots k_p \dot{0} - k'_q \ldots - k'_1] \\ (k) \qquad (k') & (\max) \end{matrix} \right\rangle = \bar{A} \bar{P}_{(k)(k')}(\eta_{is}) |0\rangle \tag{5.4}$$

transform irreducibly under the groups $U(p) \supset U(p-1) \supset \ldots \supset U(1)$ and $U(q) \supset U(q-1) \supset \ldots \supset U(1)$ and are characterised by $U(p)$ and $U(q)$ Gel'fand patterns, (k) and (k'), respectively (Gel'fand and Tseitlin 1950, Baird and Biedenharn 1963, Moshinsky 1963).

The remaining basis states of the $U(p, q)$ irrep $[k_1 \ldots k_p; k'_1 \ldots k'_q]$ can be obtained from the set of states (5.4) by applying polynomials in the operators $D^\dagger_{ij}$. A set of such linearly independent polynomials is provided by the polynomials $P^s_{(h^s)(h^{s'})}(D^\dagger_{ij})$, corresponding to all possible $U(p) \times U(q)$ irreps $[h^s_1 \ldots h^s_q \dot{0}] \times [h^s_1 \ldots h^s_q]$ and all possible Gel'fand patterns (h^s), $(h^{s'})$ of the latter. They are associated with the following basis states of the $U(p, q)$ irrep $[\dot{0}; \dot{0}]$:

$$\left| \begin{matrix} [\dot{0}; \dot{0}] & [h^s_1 \ldots h^s_q \dot{0}][\dot{0} - h^s_q \ldots - h^s_1] \\ [h^s_1 \ldots h^s_q \dot{0}][h^s_1 \ldots h^s_q]; & [\dot{0}] \\ (h^s) \qquad (h^{s'}) & \end{matrix} \right\rangle = A^s P^s_{(h^s)(h^{s'})}(D^\dagger_{ij}) |0\rangle. \tag{5.5}$$

The explicit form of $P^s_{(h^s)(h^{s'})}(D^\dagger_{ij})$ can be found by applying appropriate lowering operators to the HWS polynomial defined in (4.4).

To obtain a set of linearly independent solutions of (3.14), we only have to couple the polynomials $\bar{P}_{(k)(k')}(\eta_{is})$ and $P^s_{(h^s)(h^{s'})}(D^\dagger_{ij})$ to a definite $U(p) \times U(q)$ irrep $[h_1 \ldots h_p] \times [h'_1 \ldots h'_q]$ by means of appropriate $SU(p)$ and $SU(q)$ Wigner coefficients. For such purposes, we assume that a solution to the $SU(m)$ external state labelling problem is known for $m < n$. Hence the relevant $SU(p)$ and $SU(q)$ Wigner coefficients can, at least in principle, be evaluated. Each one of them involves a set of additional labels which we denote by χ and χ', respectively. The result is

$$\left| \begin{matrix} [k_1 \dots k_p; k'_1 \dots k'_q] & [h_1 \dots h_p \dot{0}][\dot{0} - h'_q \dots - h'_1] \\ \omega[h_1 \dots h_p][h'_1 \dots h'_q]; & \omega[k_1 \dots k_p \dot{0} - k'_q \dots - k'_1] \\ (\max) \quad (\max) & (\max) \end{matrix} \right\rangle$$

$$= A_\omega \sum_{(k)(k')(h^s)(h^{s'})} \left\langle \begin{matrix} [k_1 \dots k_p] \\ (k) \end{matrix}, \begin{matrix} [h^s_1 \dots h^s_q \dot{0}] \\ (h^s) \end{matrix} \middle| \begin{matrix} [h_1 \dots h_p] \\ (\max) \end{matrix}; \chi \right\rangle$$

$$\times \left\langle \begin{matrix} [k'_1 \dots k'_q] \\ (k') \end{matrix}, \begin{matrix} [h^s_1 \dots h^s_q] \\ (h^{s'}) \end{matrix} \middle| \begin{matrix} [h'_1 \dots h'_q] \\ (\max) \end{matrix}; \chi' \right\rangle P^s_{(h^s)(h^{s'})}(D^\dagger_{ij})$$

$$\times \left| \begin{matrix} [k_1 \dots k_p; k'_1 \dots k'_q] & [k_1 \dots k_p \dot{0}][\dot{0} - k'_q \dots - k'_1] \\ [k_1 \dots k_p][k'_1 \dots k'_q]; & [k_1 \dots k_p \dot{0} - k'_q \dots - k'_1] \\ (k) \quad (k') & (\max) \end{matrix} \right\rangle \tag{5.6}$$

where the set of additional labels ω is given by

$$\omega = [h^s_1 \dots h^s_q]\chi\chi'. \tag{5.7}$$

Since the partitions $[h^s_1 \dots h^s_q]$ are precisely those appearing in King's formula (2.12) for the multiplicities, and moreover χ and χ' solve the state labelling problems for the products $[k_1 \dots k_p] \times [h^s_1 \dots h^s_q \dot{0}]$ and $[k'_1 \dots k'_q] \times [h^s_1 \dots h^s_q]$ of $U(p)$ and $U(q)$ irreps, respectively, the number of the states (5.6), corresponding to all possible sets ω, agrees with that predicted by King's rule. We have therefore found a complete set of simultaneous solutions of (3.14), thereby solving the $SU(n)$ state labelling problem for the product of p positive-row with q negative-row irreps whenever $p+q \leqslant n$.

It is easy to check that on the right-hand side of (5.7), the number of independent labels is equal to $(p-1) \times (q-1)$, in accordance with (2.16). The q quantum numbers $h^s_1, \dots, h^s_q$ are indeed linked by the relation

$$\sum_\beta h^s_\beta = \sum_\alpha (h_\alpha - k_\alpha) = \sum_\beta (h'_\beta - k'_\beta) \tag{5.8}$$

leaving $q-1$ independent labels. Moreover, a calculation similar to the (2.16) one shows that χ and χ' contain $\frac{1}{2}(q-1)(2p-q-2)$ and $\frac{1}{2}(q-1)(q-2)$ independent labels, respectively.

Since the set of labels (5.7) cannot be associated directly with eigenvalues of Hermitian operators, the states (5.6) corresponding to the same irreps $[h_1 \dots h_p \dot{0}]$, $[\dot{0} - h'_q \dots - h'_1]$ and $[k_1 \dots k_p \dot{0} - k'_q \dots - k'_1]$, but to different sets ω and ω', are not orthogonal. Their overlap, as well as the normalisation coefficient A_ω, can be obtained by solving a recursion relation derived from a coherent state representation of $U(p, q)$ (Quesne 1987). One might then proceed to construct an orthonormal basis by following a prescription recently used for other chains of groups (Le Blanc and Rowe 1985, Hecht 1986), or any other procedure.

It now remains to extend the solutions (5.6) of (3.14) to the case where $p+q > n$ to obtain a complete solution to the $SU(n)$ external state labelling problem. Such a generalisation is outlined in the next section.

6. The general case for $p+q>n$

Whenever $p+q>n$, we may still use the two complementary chains (3.13a) and (3.13b), but the irreps of the complementary groups $U(n)$ and $U(p, q)$ are to be modified as follows (Quesne 1986):

$$\begin{array}{ccccc} [h_1 \ldots h_p\dot{0}] & [\dot{0}-h'_q \ldots -h'_1] & [k_1 \ldots k_{n-q+\sigma}-k'_{q-\sigma} \ldots -k'_1] \\ \mathrm{U}(n) & \times & \mathrm{U}(n) & \supset & \mathrm{U}(n) \end{array} \tag{6.1a}$$

$$\begin{array}{ccccc} \mathrm{U}(p) & \times & \mathrm{U}(q) & \subset & \mathrm{U}(p,q) \\ [h_1 \ldots h_p] & & [h'_1 \ldots h'_q] & & [k_1 \ldots k_{n-q+\sigma}\dot{0};\, k'_1 \ldots k'_{q-\sigma}\dot{0}] \end{array} \tag{6.1b}$$

Here $k_1, \ldots, k_{n-q+\sigma}, k'_1, \ldots, k'_{q-\sigma}$ are non-negative integers satisfying the conditions $k_1 \geqslant \ldots \geqslant k_{n-q+\sigma}$, $k'_1 \geqslant \ldots \geqslant k'_{q-\sigma}$, and σ is any integer belonging to the set $\{0, 1, \ldots, p+q-n\}$. For a generic U(n) irrep $[k_1 \ldots k_{n-q+\sigma} - k'_{q-\sigma} \ldots - k'_1]$, the number of needed additional labels is now given by

$$\tfrac{1}{2}p(2n-p-1)+\tfrac{1}{2}q(2n-q-1)-\tfrac{1}{2}(n-1)(n+2) \tag{6.2}$$

instead of $(p-1)(q-1)$, as in (2.16).

Equations (3.14a), (3.14b) and (3.14d) remain unchanged, while (3.14c) is replaced by

$$\left(\sum_{i=1}^{p} \eta_{is}\xi_{is} - \sum_{i=p+1}^{d} \eta_{is}\xi_{is}\right) P(\eta_{is})|0\rangle$$

$$= \begin{cases} k_s P(\eta_{is})|0\rangle & s = 1, \ldots, n-q+\sigma \\ -k'_{n+1-s} P(\eta_{is})|0\rangle & s = n-q+\sigma+1, \ldots, n. \end{cases} \tag{6.3}$$

A set of simultaneous solutions of this modified system of equations can be written as

$$\left| \begin{array}{cc} [k_1 \ldots k_{n-q+\sigma}\dot{0};\, k'_1 \ldots k'_{q-\sigma}\dot{0}] & [h_1 \ldots h_p\dot{0}][\dot{0}-h'_q \ldots -h'_1] \\ \omega[h_1 \ldots h_p][h'_1 \ldots h'_q] \quad ; & \omega[k_1 \ldots k_{n-q+\sigma}-k'_{q-\sigma} \ldots -k'_1] \\ (\max) \qquad (\max) & (\max) \end{array} \right\rangle$$

$$= A_\omega \sum_{(k)(k')(h^s)(h^{s'})} \left\langle \begin{array}{cc} [k_1 \ldots k_{n-q+\sigma}\dot{0}] \, , & [h^s_1 \ldots h^s_q\dot{0}] \\ (k) & (h^s) \end{array} \middle| \begin{array}{c} [h_1 \ldots h_p] \\ (\max) \end{array} ; \chi \right\rangle$$

$$\times \left\langle \begin{array}{cc} [k'_1 \ldots k'_{q-\sigma}\dot{0}] \, , & [h^s_1 \ldots h^s_q] \\ (k') & (h^{s'}) \end{array} \middle| \begin{array}{c} [h'_1 \ldots h'_q] \\ (\max) \end{array} ; \chi' \right\rangle P^s_{(h^s)(h^{s'})}(D^\dagger_{ij})$$

$$\times \left| \begin{array}{cc} [k_1 \ldots k_{n-q+\sigma}\dot{0};\, k'_1 \ldots k'_{q-\sigma}\dot{0}] & [k_1 \ldots k_{n-q+\sigma}\dot{0}][\dot{0}-k'_{q-\sigma} \ldots -k'_1] \\ [k_1 \ldots k_{n-q+\sigma}\dot{0}][k'_1 \ldots k'_{q-\sigma}\dot{0}]; & [k_1 \ldots k_{n-q+\sigma}-k'_{q-\sigma} \ldots -k'_1] \\ (k) \qquad (k') & (\max) \end{array} \right\rangle \tag{6.4}$$

where the set of additional labels ω is still given by (5.7). Since χ and χ' now contain $\frac{1}{2}(n-q+\sigma)(2p+q-n-\sigma-1)+\frac{1}{2}q(2p-q-1)-\frac{1}{2}(p-1)(p+2)$ and $\frac{1}{2}(q-\sigma)(q+\sigma-1)-(q-1)$ independent labels, ω provides

$$\tfrac{1}{2}p(2n-p-1)+\tfrac{1}{2}q(2n-q-1)-\tfrac{1}{2}(n-1)(n+2)+\sigma(p+q-n-\sigma) \tag{6.5}$$

independent additional labels. Comparison with (6.2) shows that the latter number exceeds that of needed labels by $\sigma(p+q-n-\sigma)$. Hence, whenever σ is different from 0 and $p+q-n$, ω contains some redundant independent labels.

As a counterpart, in general the states (6.4) are not linearly independent as it was the case for the states (5.6). This is a consequence of the modification rule which makes some of the irreps allowed by the rules (2.11) and (2.12) disappear. Methods developed for other chains of groups (Quesne 1984, Le Blanc and Rowe 1985, Hecht and Elliott 1985, Hecht 1985, 1986) could be used here to obtain the relations between the states (6.4).

7. Conclusion

For the most general SU(n) irreps (corresponding to $p=q=n-1$), the solution to the SU(n) external state labelling problem, as proposed in the present paper, is based upon the group chain $\mathrm{U}(n-1,n-1)\supset\mathrm{U}(n-1)\times\mathrm{U}(n-1)$. The additional labels ω include those of an intermediate U($n-1$) irrep $[h_1^s\ldots h_{n-1}^s]$, as well as the additional labels χ and χ', solving the state labelling problems for the products $[k_1\ldots k_{n-1}]\times[h_1^s\ldots h_{n-1}^s]$ and $[k_1'\ldots k_{n-1}']\times[h_1^s\ldots h_{n-1}^s]$ of U($n-1$) irreps. We have therefore obtained a recursive solution to the SU(n) external state labelling problem: provided we know such a solution for SU($n-1$), we are able to construct it for SU(n). This solution reflects in a direct way the operation of King's branching rule for the chain $\mathrm{U}(n)\times\mathrm{U}(n)\supset\mathrm{U}(n)$, supplemented, whenever necessary, with King's modification rule.

The appearance of pseudo-unitary groups in the present context is not surprising, since it has recently been shown (Deenen and Quesne 1986) that the SO(6, 2) model of SU(3) (Biedenharn and Flath 1984, Bracken and MacGibbon 1984) can be reformulated in terms of a U(1, 1) group and extended to a family of $n-2$ models of SU(n) by using higher-dimensional U(p, q) groups. The SO(6, 2) model providing a global algebraic formulation of the canonical SU(3) tensor operator structure, one may suspect that the present solution to the SU(n) external state labelling problem is related to that of Baird and Biedenharn (1964, 1965). We plan to review this point in a forthcoming paper of the present series.

References

Baird G E and Biedenharn L C 1963 *J. Math. Phys.* **4** 1449
—— 1964 *J. Math. Phys.* **5** 1730
—— 1965 *J. Math. Phys.* **6** 1847
Biedenharn L C and Flath D E 1984 *Commun. Math. Phys.* **93** 143
Biedenharn L C, Lohe M A and Louck J D 1985 *J. Math. Phys.* **26** 1458
Bracken A J and MacGibbon J H 1984 *J. Phys. A: Math. Gen.* **17** 2581
Brody T A, Moshinsky M and Renero I 1965 *J. Math. Phys.* **6** 1540
Deenen J and Quesne C 1983 *J. Phys. A: Math. Gen.* **16** 2095
—— 1986 *J. Phys. A: Math. Gen.* **19** 3463
Draayer J P and Akiyama Y 1973 *J. Math. Phys.* **14** 1904
Flores J 1967 *J. Math. Phys.* **8** 454
Flores J and Moshinsky M 1967 *Nucl. Phys.* A **93** 81
Gelbert S 1979 *Proc. Symp. Pure Math. Am. Math. Soc.* **33** 287
Gel'fand I M and Tseitlin M L 1950 *Dokl. Akad. Nauk* **71** 825
Gross K and Kunze R 1977 *J. Funct. Anal.* **25** 1
Hecht K T 1965 *Nucl. Phys.* **62** 1
—— 1985 *Nucl. Phys.* A **444** 189

—— 1986 *Proc. 9th Symp. on Nuclear Physics, Oaxtepec, Mexico*
Hecht K T and Elliott J P 1985 *Nucl. Phys.* A **438** 29
Howe R 1979 *Proc. Symp. Pure Math. Am. Math. Soc.* **33** 275
Kashiwara M and Vergne M 1978 *Invent. Math.* **44** 1
King R C 1970 *J. Math. Phys.* **11** 280
—— 1971 *J. Math. Phys.* **12** 1588
King R C and Wybourne B G 1985 *J. Phys. A: Math. Gen.* **18** 3113
Le Blanc R and Rowe D J 1985 *J. Phys. A: Math. Gen.* **18** 1891
Littlewood D E 1950 *The Theory of Group Characters* (Oxford: Clarendon)
Littlewood D E and Richardson A R 1934 *Phil. Trans. R. Soc.* A **233** 99
Moshinsky M 1962 *Rev. Mod. Phys.* **34** 813
—— 1963 *J. Math. Phys.* **4** 1128
Moshinsky M and Quesne C 1970 *J. Math. Phys.* **11** 1631
—— 1971 *J. Math. Phys.* **12** 1772
Moshinsky M and Syamala Devi V 1969 *J. Math. Phys.* **10** 455
Nagel J G and Moshinsky M 1965 *J. Math. Phys.* **6** 682
Newell M J 1951 *Proc. R. Irish Acad.* A **54** 143, 153
Quesne C 1984 *J. Phys. A: Math. Gen.* **17** 777, 791
—— 1985a *J. Phys. A: Math. Gen.* **18** 1167
—— 1985b *J. Phys. A: Math. Gen.* **18** 2675
—— 1986 *J. Phys. A: Math. Gen.* **19** 2689
—— 1987 *J. Phys. A: Math. Gen.* **20** 785
Seligman T H and Sharp R T 1983 *J. Math. Phys.* **24** 769
Sharp R T and Lam C S 1969 *J. Math. Phys.* **10** 2033